Calculation of spin resonance harmonic. Snake resonances.

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Calculation of the spin resonance harmonics with the Snakes

- The DEPOL code which is usually used to calculate the spin resonance strengths, always assumes the vertical spin everywhere on the design orbit.

- Addressing the question from last year MAC review, another code has been written to calculate the spin resonance harmonics in an accelerator with an arbitrary configuration of the stable spin on the design orbit. Thus, the calculation for RHIC can be done with full configuration of the Snakes and spin rotators.

Calculating the spin resonance harmonics

$$\dot{\vec{S}} = \left(\vec{W}_0 + \vec{w}\right) \times \vec{S}$$

$\vec{n}_0, \vec{\eta}_1, \vec{\eta}_2$ - constructed from the eigenvectors of one-turn map, on the basis of $\mathbf{W_0}$; provide spin-related coordinate frame

$$\vec{\eta} = \vec{\eta}_1 - i\vec{\eta}_2, \qquad \vec{\eta}\left(2\pi\right) = \vec{\eta}\left(0\right)e^{i2\pi\nu_{sp}}$$

$$w_k = \frac{1}{2\pi}\int_0^{2\pi} (\vec{w}\vec{\eta})_{\nu_k}\, e^{i(\nu_k - \nu_{sp})\theta}\, d\theta$$ - resonance circular harmonic in general case

In this presentation: x-horizontal, y- longitudinal, z-vertical

Perturbation spin precession vector:
(only terms proportional to ν_0 are kept, high energy accelerator)

$$\nu_0 = G\gamma$$

$$w_x = \nu_0 z'' + \nu_0 K_x \frac{\Delta p}{p}$$

$$w_y = -\nu_0 (K_x x' + K_z z')$$

$$w_z = -\nu_0 x'' + \nu_0 K_z \frac{\Delta p}{p}$$

K_x and K_z – normalized horizontal and vertical bending fields

The code features

•The calculation of the resonances is based on the integrals over the one turn.
•The corresponding code has been added into the ASPIRRIN program (which calculates the spin-orbital functions).

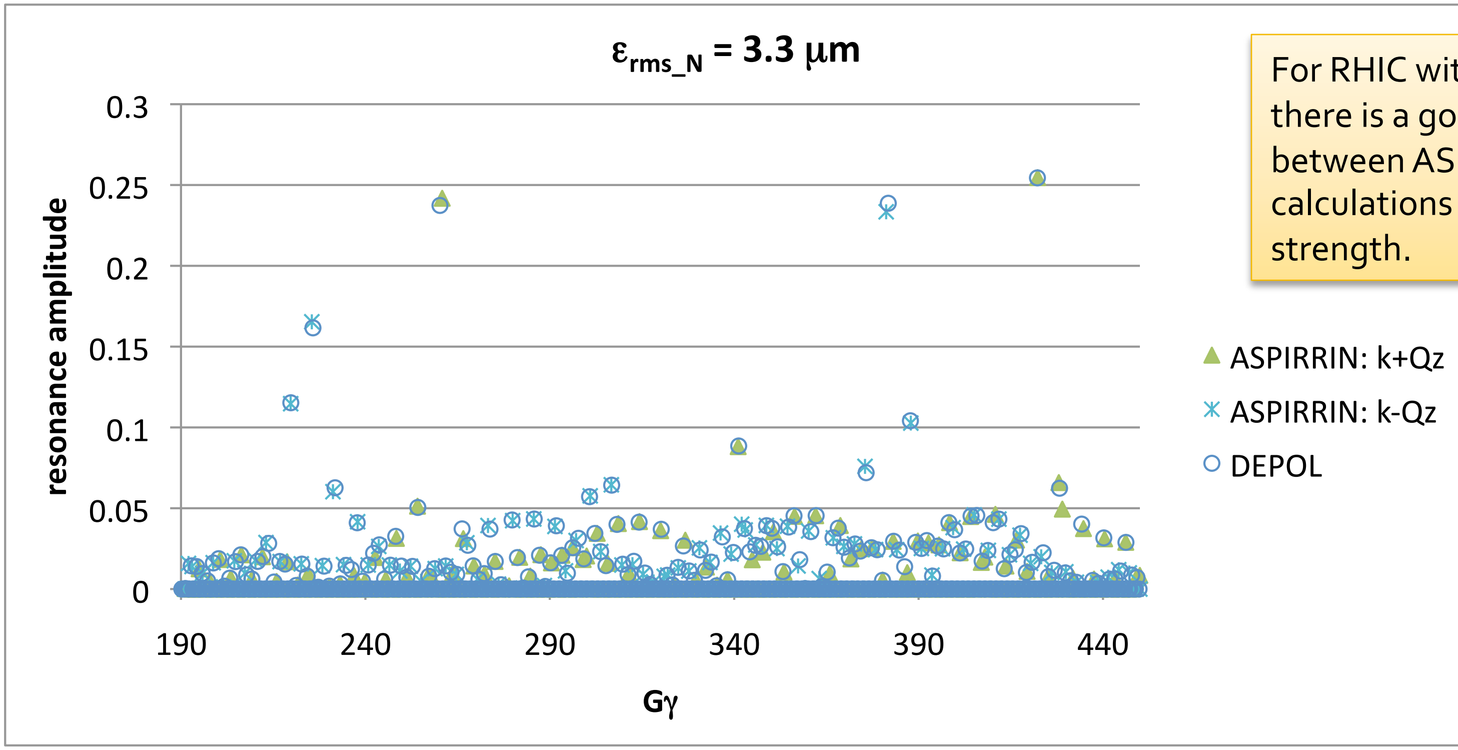
•The ASPIRRIN contains inside the full 4-D calculation of the coupled betatron motion, so, ultimately, the resonance calculation could be done in the presence of the betatron coupling.

•But, so far, only the main effect, originating from the quadrupole magnets has been included (no betatron coupling yet).

•The configuration of the spin stable direction on the design orbit can be arbitrary: with Snakes and rotators.
Snakes and rotators are approximated by the spin rotation transformation.

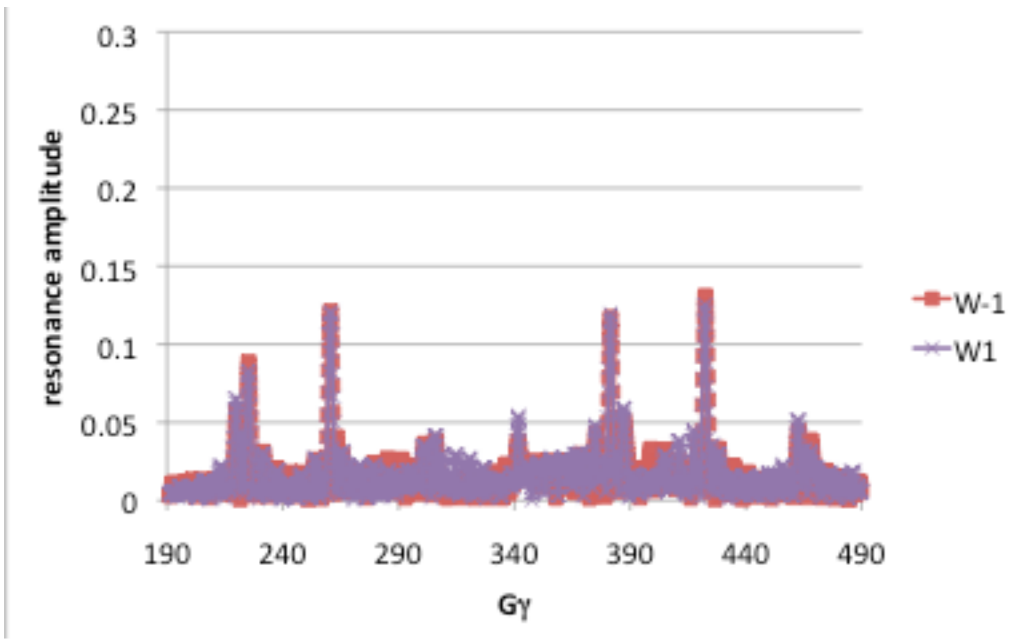
•Lattice input is done from MAD output twiss file.

RHIC without Snakes

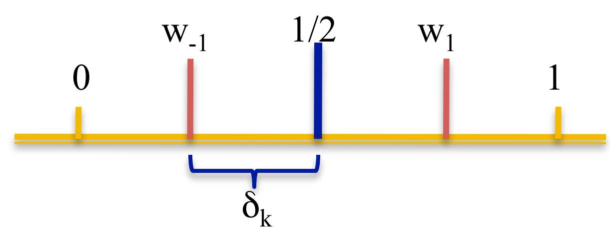


For RHIC without Snakes there is a good agreement between ASPIRRIN and DEPOL calculations of resonance strength.

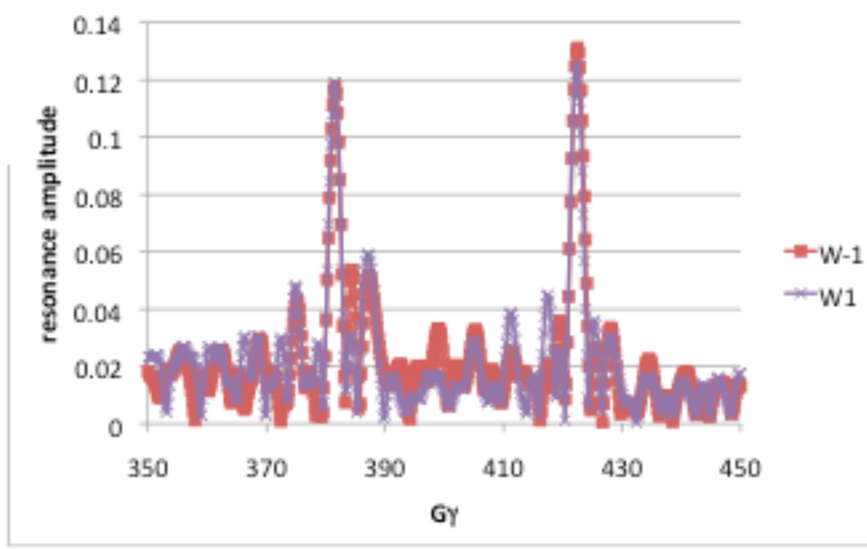
RHIC with 2 Snakes



The plots show two circular resonance harmonics closest to the spin tune ½.
w_{-1} -> harmonic 30 $-Q_z$
w_1 -> harmonic -29+Q_z



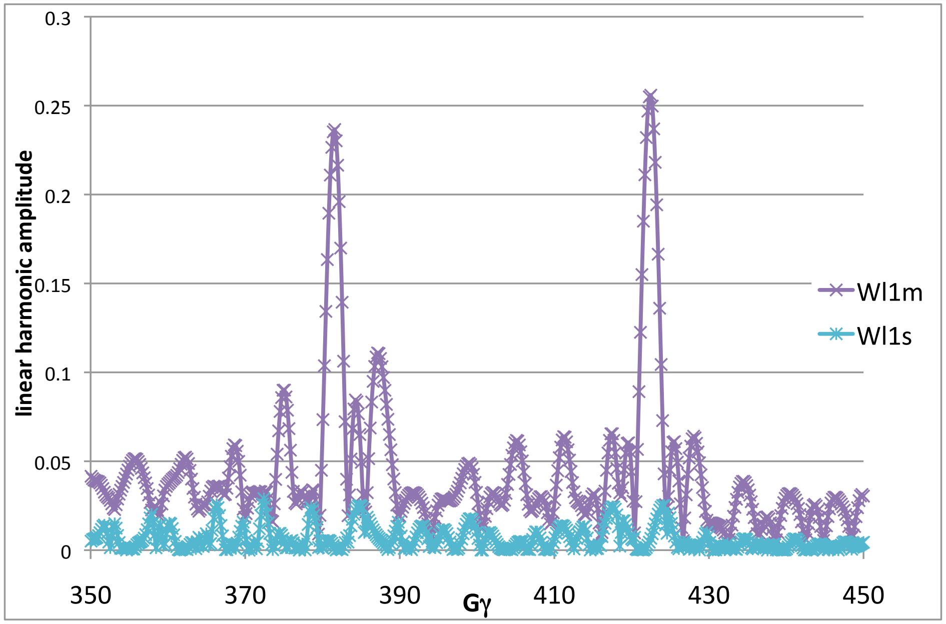
Zoomed in on the two strong resonance region.



The amplitudes of circular harmonics are often near equal. Especially at strong resonance. Standard picture of one isolated (circular harmonic) resonance, can not be applied here.

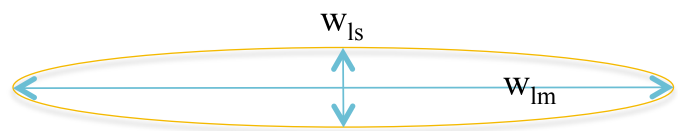
Linear harmonics consideration is more adequate in the case on the accelerator with Snakes.

Presentation in linear harmonics



In most Gγ areas: $|w_{l1m}| >> |w_{l1s}|$

Main and secondary linear harmonics definitions:



$$w_{l1m} = \left(|w_1| + |w_{-1}|\right)\cdot e^{i\alpha}\cos(\delta_1\theta + \varphi_0)$$

$$w_{l1s} = i\left(|w_1| - |w_{-1}|\right)\cdot e^{i\alpha}\sin(\delta_1\theta + \varphi_0)$$

$\delta_1=\{Q_z\}-1/2$

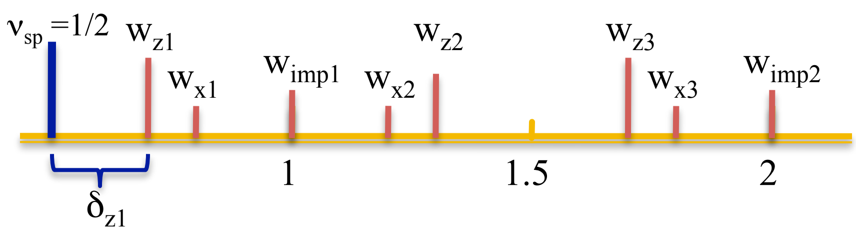
The solution for spin vector n for the case of one linear harmonic can be easily found:

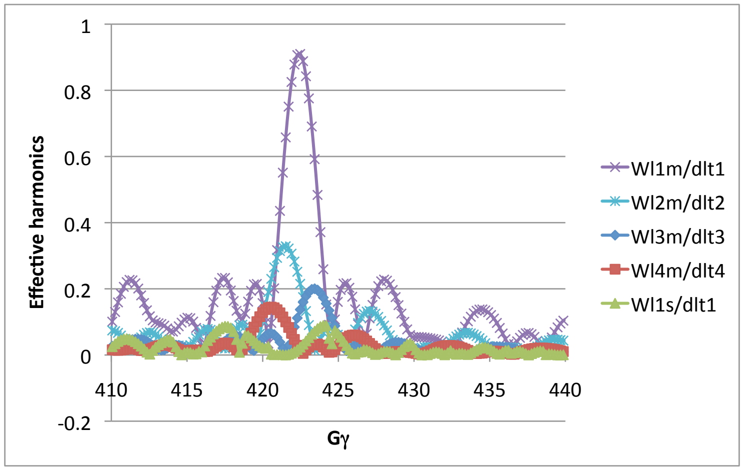
$$n_x = 0, \quad n_y = -\sin\rho, \quad n_z = \cos\rho$$

Describes the first-order resonance $\{Q_z\} \sim 1/2$

$$\rho = \frac{|w_{l1m}|}{\delta_1}\cos(\delta_1\theta + \varphi_0)$$

Snake resonances





The plot demonstrates relative effect of different harmonics on the spin motion, evaluated as $|w_k|/\delta_k$ Only harmonics related with Q_z are shown.

The effect of the other harmonics, located further away from the spin tune, $\delta_k > \delta_{z1}$, should be included by the methods of perturbation theory.

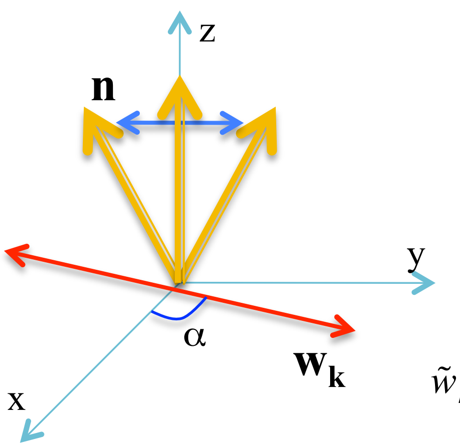
This harmonic interactions lead to the high-order (Snake) resonances in most general form:

$$\frac{1}{2} = mQ_z + nQ_x + k$$

Example of perturbation approach

There are several perturbation theory approaches considered for obtaining higher-order spin solutions. Here is the example of one possible approach.

We consider the distortion of the solution shown before for the main linear harmonic related with Q_z, by another linear harmonic w_k



$$n_x = 0, \quad n_y = -\sin\rho, \quad n_z = \cos\rho$$

$$\rho = \frac{|w_1|}{\delta_1}\cos(\delta_1\theta + \varphi_0), \quad \delta_1 = \{Q_z\} - 1/2$$

$$w_k = |w_k| \cdot \cos(\delta_k\theta + \varphi_k)$$

$$\delta_k = \nu_k - 1/2$$

After transformation to the system where the spin solution **n** is constant, the orthogonal to **n** part of the perturbation will be:

$$\tilde{w}_{k\perp} = |w_k|\sin\alpha\cos(\delta_k\theta + \varphi_k)\cos\rho = |w_k|\sin\alpha \cdot \cos(\delta_k\theta + \varphi_k)\sum_p \left[(-1)^p J_{2p}\left(\frac{|w_1|}{\delta_1}\right) e^{i2p(\delta_1\theta + \varphi_0)}\right]$$

From here, the general resonance condition is: $\nu_k \pm 2pQ_z = \frac{1-2p}{2}$

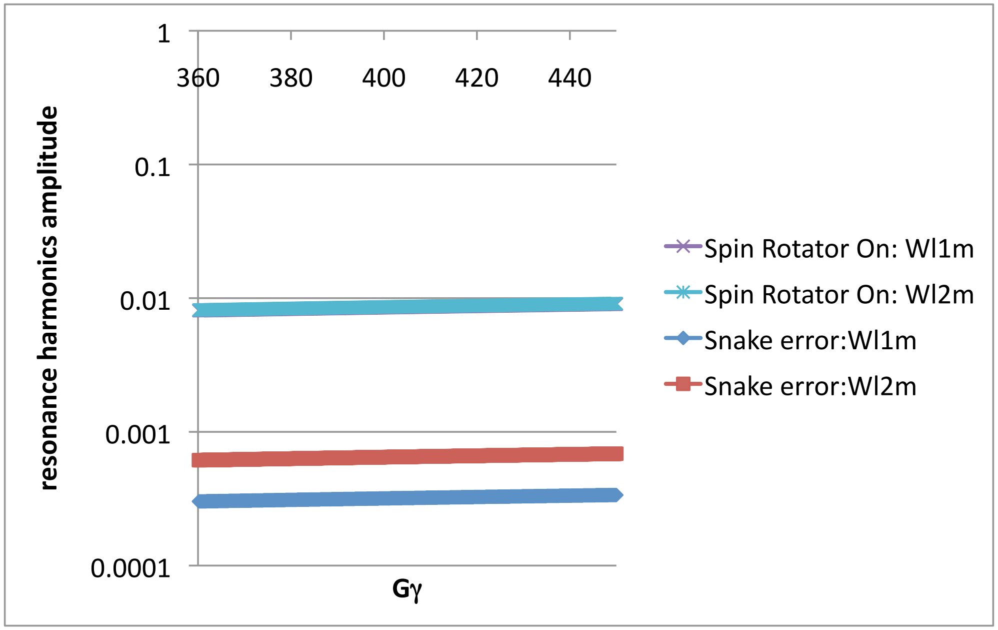
And the resonance strength: $|w_k|\sin\alpha \cdot J_{2p}\left(\frac{|w_1|}{\delta_1}\right)$

$$\boxed{\begin{array}{l} \nu_k = \pm Q_z + k \\ Q_z = \pm\dfrac{1-2(p+k)}{2(2p\pm 1)} \end{array}}$$

$$\boxed{\begin{array}{l} \nu_k = k \\ Q_z = \pm\dfrac{1-2(p+k)}{4p} \end{array}}$$

$$\boxed{\begin{array}{l} \nu_k = \pm Q_x + k \\ Q_x \pm 2pQ_z = \pm\dfrac{1-2(p+k)}{2} \end{array}}$$

Example of horizontal harmonics



Horizontal resonance harmonics can appear if:

•vector n_0 on closed orbit deviates from the vertical (due to imperfection resonances)

•betatron coupling

No secondary linear harmonics: $w_{xl1s}=w_{xl2s}=0$

First case: IP6 spin rotator turned on (with energy independent settings)

Second case: 5^o spin rotation angle error of 5^o in 9 o'clock snake

In RHIC the interaction of the horizontal and vertical harmonics may be important because of the closeness of betatron tunes (Q_z-Q_x~0.01)

SRM - single resonance model

The important case is the "single resonance model", or SRM, where single resonance (circular) harmonic υ_n, defined in the accelerator without the Snakes, is considered together with the Snake spin transformations.

Single resonance (circular) harmonic υ_n is transformed in the accelerator with the Snakes into the orthogonal series of linear harmonics (so, the "single" resonance is not really single in the proper frame) :

$$\upsilon_n e^{i\nu_n\theta} \to -e^{i\tilde{\alpha}} \sum_k i^k w_{nk} \sin(\delta_k \theta + \phi_n - \xi)$$

$$w_{nk} = |\upsilon_n| \frac{\sin(\chi_k \pi/2)}{\chi_k \pi/2}, \quad \delta_k = \nu_n - (k+1/2),$$

$$\chi_k = \nu_0 - (k+1/2), \quad \tilde{\alpha} = \xi_1 - \nu_0 \tfrac{\pi}{2} - \tfrac{\pi}{4}$$

•The axis of the linear harmonics does not depend on the harmonic phase.
•w_{nk} presents main linear harmonics. There are no secondary ones.
Since in RHIC $|w_{lm}| >> |w_{ls}|$ one can expect the SRM can be applicable in wide ranges of Gγ, and especially near the strong resonances.
•<u>The best thing: the analytical solution for the SRM with the Snakes was obtained by S. Mane.</u>

Mane's solution for Snake resonances for SRM

S. Mane derived the analytical solutions for the SRM with the Snakes. For instance, for the case of two Snakes the solution for the periodical spin vector field n is:

$$n_z = \mathscr{A}_0(\eta, \pi\delta) + 2 \sum_{m=2,4,6,\ldots} \mathscr{A}_m(\eta, \pi\delta) \cos\left(m(\phi - \xi)\right)$$

$$n_x + i n_y = -2ig \sum_{m=1,3,5,\ldots} \mathscr{B}_m(\eta, \pi\delta) \sin\left(m(\phi - \xi)\right)$$

betatron phase

Snake axis angle

$$g_2 = \left[\cos\frac{\pi\Omega}{2} + \mathrm{i}\,\frac{\nu_0 - Q}{\Omega}\sin\frac{\pi\Omega}{2}\right] \mathrm{e}^{\mathrm{i}\xi}\,\mathrm{e}^{\mathrm{i}\pi\delta/2}.$$

$$\Omega = \sqrt{(\nu_0 - Q_k)^2 + |\upsilon_k|^2}$$

$$\eta = \frac{|\upsilon_k|}{\Omega}\sin\left(\frac{\pi\Omega}{2}\right)$$

$$\delta = Q_k - \tfrac{1}{2}$$

Sine-Bessel functions:

$$\mathscr{A}_m(z,\alpha) := \cos\left(\frac{1}{2}m\alpha\right) \sum_{k=0}^{\infty} (-1)^k \frac{C^2_{m/2+k-1}(\alpha)}{S_k(\alpha) S_{k+m}(\alpha)} z^{m+2k}$$

$$\mathscr{B}_m(z,\alpha) := \sum_{k=0}^{\infty} (-1)^k \frac{C^2_{(m-1)/2+k}(\alpha)}{S_k(\alpha) S_{k+m}(\alpha)} (z\mathrm{e}^{\mathrm{i}\alpha/2})^{m+2k}.$$

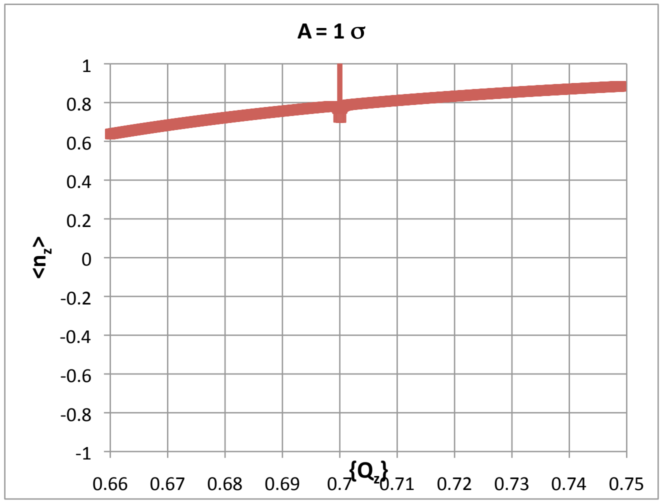
$$S_n(\alpha) := \sin\alpha \sin(2\alpha) \cdots \sin(n\alpha)$$

$$C_n(\alpha) := \cos\alpha \cos(2\alpha) \cdots \cos(n\alpha)$$

The functions $\mathscr{A}_m$ and $\mathscr{B}_m$ has the resonance behavior at the snake resonance conditions:

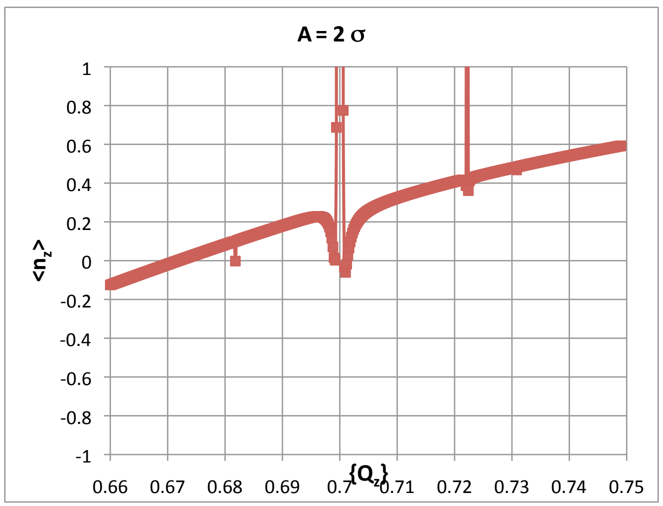
$$Q_z = \frac{2m+1}{2(2n+1)}$$

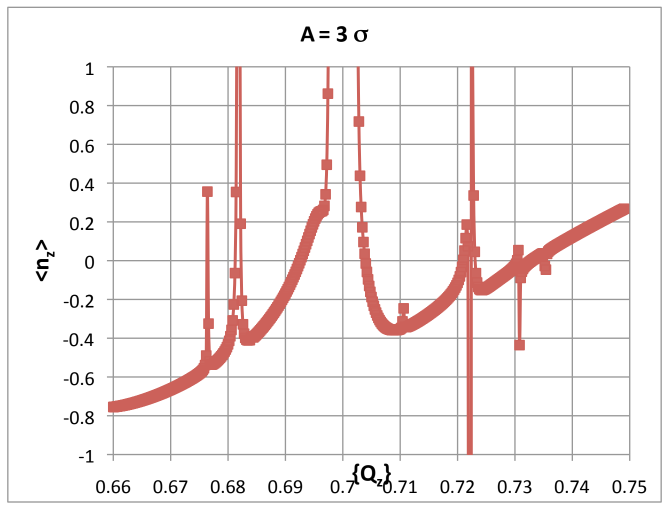
Calculation using SRM analytical solution for RHIC



Averaging over the betatron phases gives: $\langle n_z \rangle = \mathscr{A}_0(\eta, \pi\delta)$

$\langle n_z \rangle$ versus vertical betatron tune calculation are shown for the case of RHIC strongest resonances.
The plots corresponds to the vertical betatron amplitudes 1,2 and 3σ, for the normalized emittance 15 mm mrad.



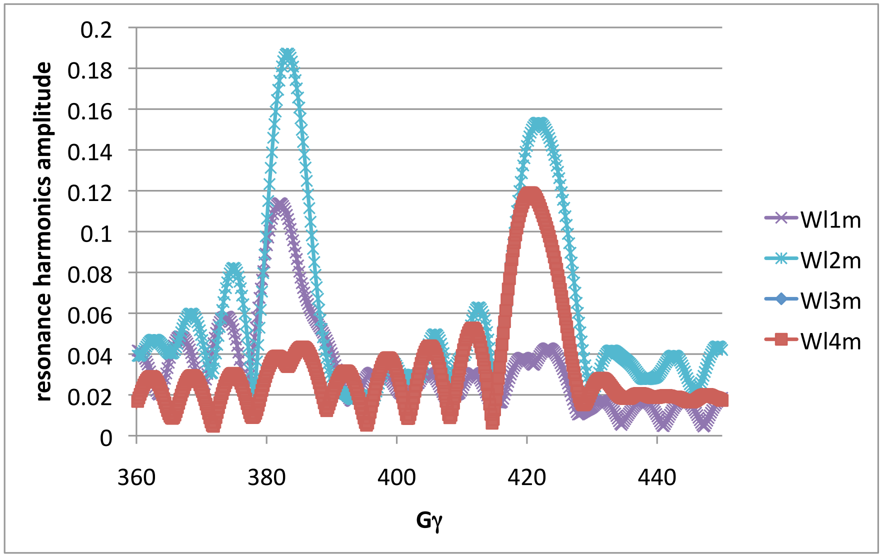


Notes on Snake resonances

- SRM analytical solution should work well in RHIC for the Snake resonances due to the interaction of Q_z-related harmonics.
- The SRM solution does not provide answer for:
  - The resonances due to the interaction of imperfection and intrinsic resonance harmonics (that is integer and Q_z related harmonics)
  - The resonances due to the interaction of the horizontal and vertical resonance harmonics
  - Due to interference of main and secondary linear resonance harmonics.
- Conditions on the loss of the adiabaticity of the spin motion near the snake resonances have to be evaluated. This is what causes the depolarization.

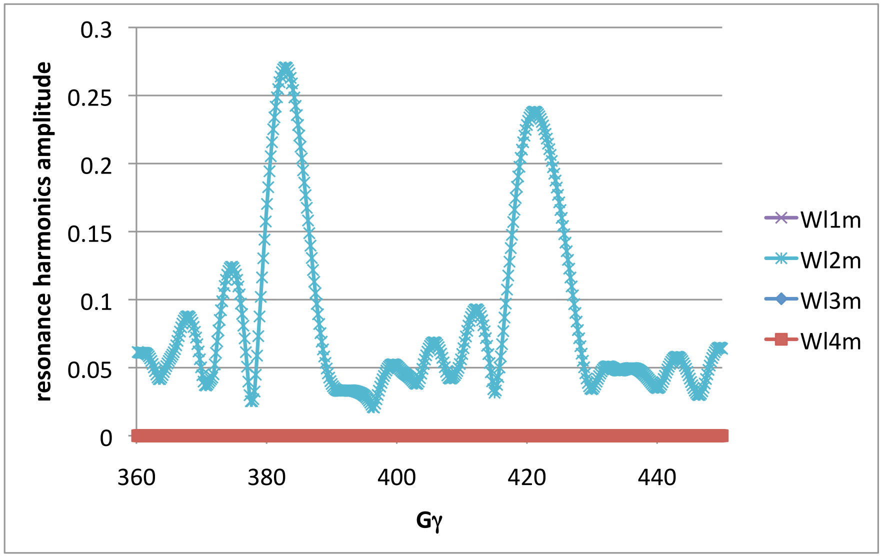
Case of six Snakes in RHIC. Different configurations.

Main linear resonance harmonics related with Q_z are shown.



Snake angle orientation: (90^o, 0^o, -45^o, +45^o, -90^o, 0^o)
All harmonics are present

Snake angle orientation: (-45^o, 45^o, -45^o, 45^o, -45^o, 45^o)
Only 2nd harmonic remains.



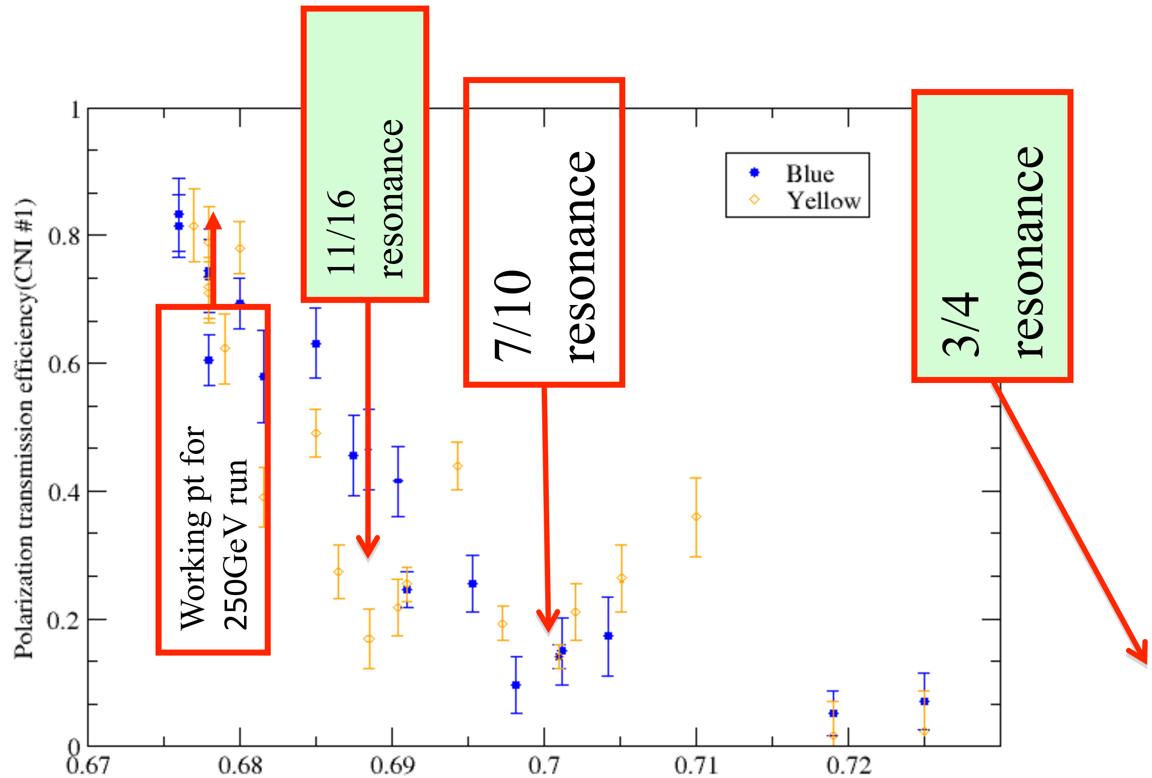
Summary

- The code was written to calculate the spin resonance harmonics in the accelerator with the arbitrary configuration of the spin on the design orbit.
- For an accelerator with the Snakes the presentation of spin perturbation in linear harmonics is more natural.
- The high-order resonances (Snake resonances) appears as result of the harmonic interaction.
- Analytical solution for SRM does exist. Can be used in RHIC for calculating the Snake resonances produced by the interaction of harmonics related with Q_z.
- Further work:
  - Evaluating of the strength of the Snake resonances due to interaction of the integer and Q_x related harmonics with Q_z-related harmonics. Using perturbation theory approaches.
  - Evaluating the conditions of the spin motion adiabaticity loss near the Snake resonances.

Backup Slides

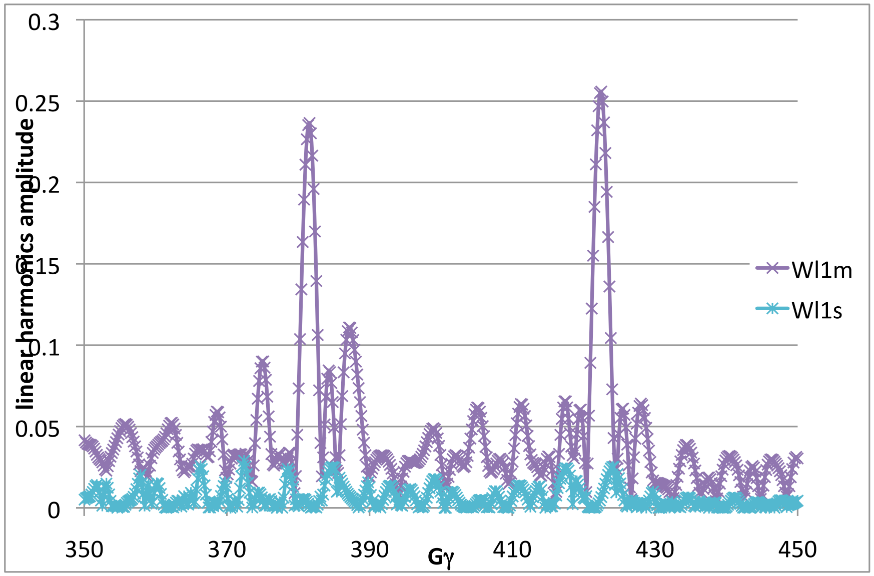
Measured snake resonances (from Run-9)

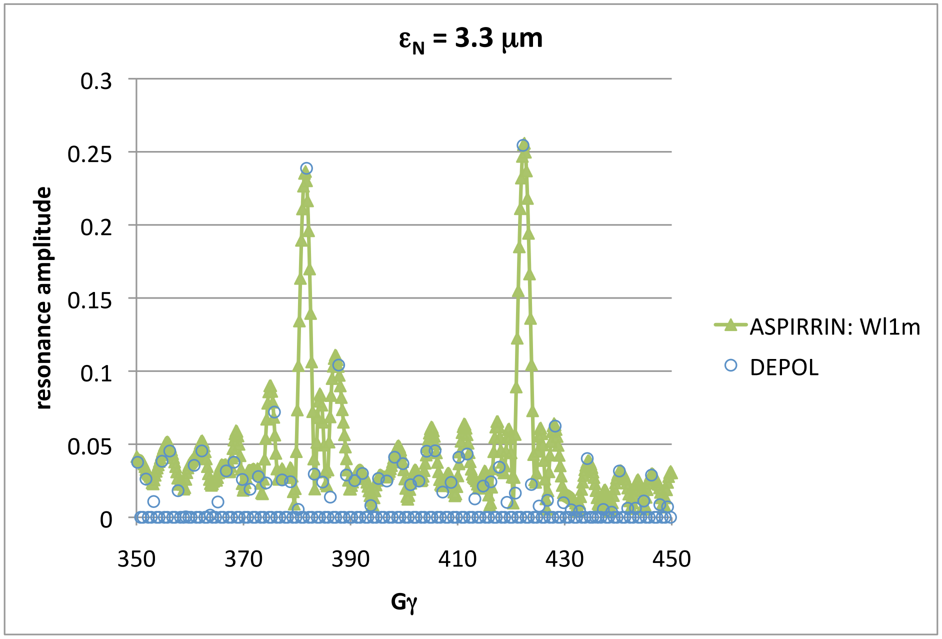
The orbit and snake errors lead to the even order resonances (light green)



•Very strong depolarization from ¾ resonance indicates large imperfection error. Most probably originating in the vertical closed orbit errors.

Presentation in linear harmonics





Snake resonances and orbit related effects

- Depolarization is caused by high-order ("snake") resonances:

$$Q_z = \frac{2m+1}{2n} \pm \frac{\delta Q_{sp}}{n}$$

- Without orbit errors: only resonances with odd n (for instance, 7/10 resonance)

- The closed orbit errors affect:
  - The spin tune shift (δQ_s) and related split and shift of the snake resonance locations.
  - The snake resonance width.

    A) excitation of even-order (even n) snake resonances (for instance ¾ resonance)

    B) enhancement of odd-order snake resonances ($\sim\pi^2|\varepsilon_{imp}|^2$).
    Important only for very strong resonances ($|\varepsilon_{imp}|$ >> 0.1)

The same effects as from the orbit errors can arise from snake imperfections.